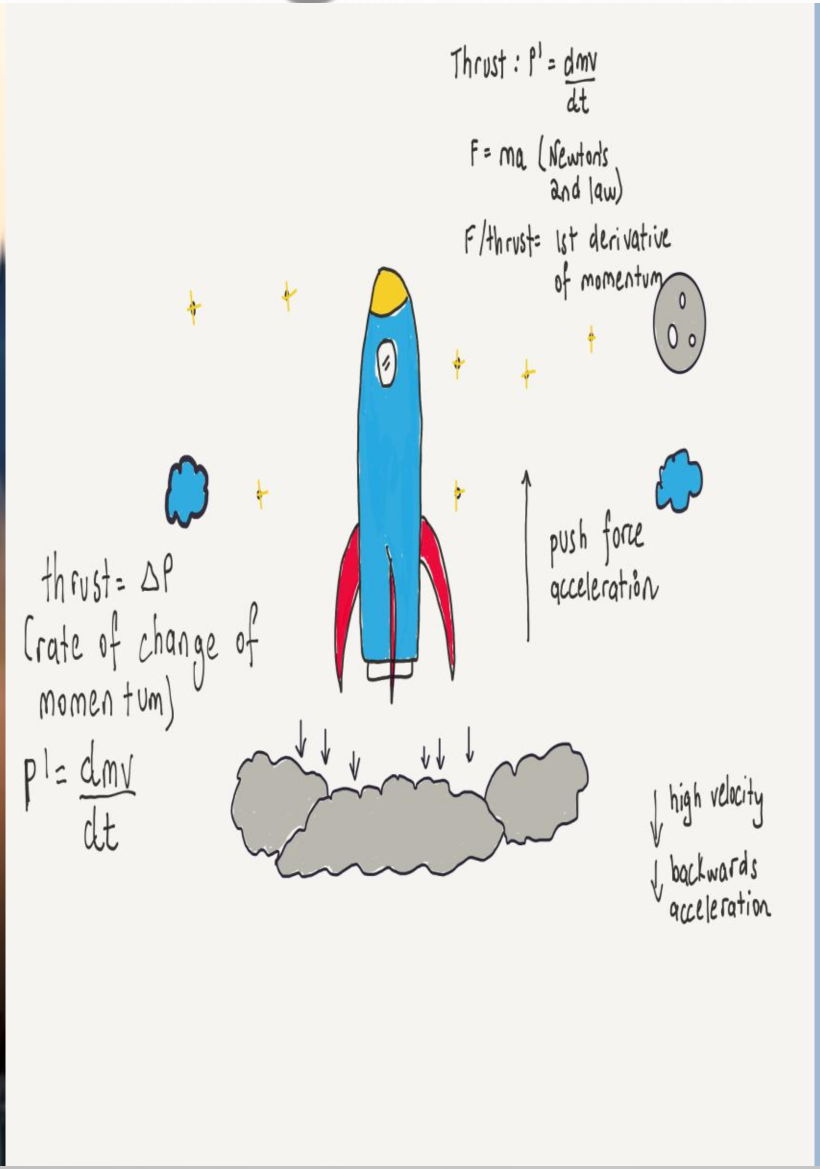
APPLICATION OF DERIVATIVES



- RATE OF CHANGE OF QUATITIES
- INCREASING &DECREASING FUNCTIONS
- TANGENTS &NORMALS
- APPROXIMATIONS
- MAXIMA &MINIMA

MODULE -1 RATE OF CHANGE OF QUATITIES

Question

The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Solution

The area of a circle (A) with radius (r) is given by,

$A = \pi r^2$

Now, the rate of change of area (A) with respect to time (t) is given by,

$\frac{dA}{dt} = \frac{d}{d}(\pi r^2).\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$ [By chain rule]

It is given that,

$\frac{dr}{dt} = 3cm/s$

$\therefore \frac{dA}{dt} = 2\pi r(3) = 6\pi r$

Thus, when $r = 10cm$,

$\frac{dA}{dt} = 6\pi(10) = 60\pi\ cm^2/s$

Hence, the rate at which the area of the circle is increasing when the radius is 10 cm is $60\pi\ cm^2/s$.

$A = \pi r^2$
HERE , A and r are the variables to be differentiated w.r.t time t

Question :

A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Solution :

The area of a circle (A) with radius (r) is given by

$A = \pi r^2$

Therefore, the rate of change of area (A) with respect to time (t) is given by,

$$\therefore \frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\frac{dr}{dt} = 2\pi r\frac{dr}{dt} \qquad \text{(by chain rule)}$$

It is given that $\frac{dr}{dt} = 5\,cm/s$

Thus, when $r = 8cm$,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi$$

Hence, when the radius of the circular wave is $8\,cm$, the enclosed area is increasing at the rate of $80\pi\ cm^2/s$.

Waves speed in cm/s shows distance/time .....so its dr/dt

Question

A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Solution

The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

$\therefore$ Rate of change of volume (V) with respect to time (t) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \quad \text{(by chain rule)}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

It is given that

$$\frac{dv}{dt} = 900\ cm^3 / S$$

$$\therefore 900 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

Therefore, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi (15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi}$ cm/s

CUBIC CM IS GIVEN--- VOLUME IS GIVEN. V.OF SPHERE=$\frac{4\pi r^3}{3}$

Question

A particle moving along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x-coordinate.

y changes 8 times as fast as x

Solution

The equation of the curve is given as:

$6y = x^3 + 2$

The rate of change of the position of the particle with respect to time (t), is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the y-coordinate of the particle changes 8 times as fast as the x-coordinate i.e., $\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$, we have:

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$

$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$

$$\Rightarrow (x^2 - 16)\frac{dx}{dt} = 0$$

$$\Rightarrow x^2 = 16$$

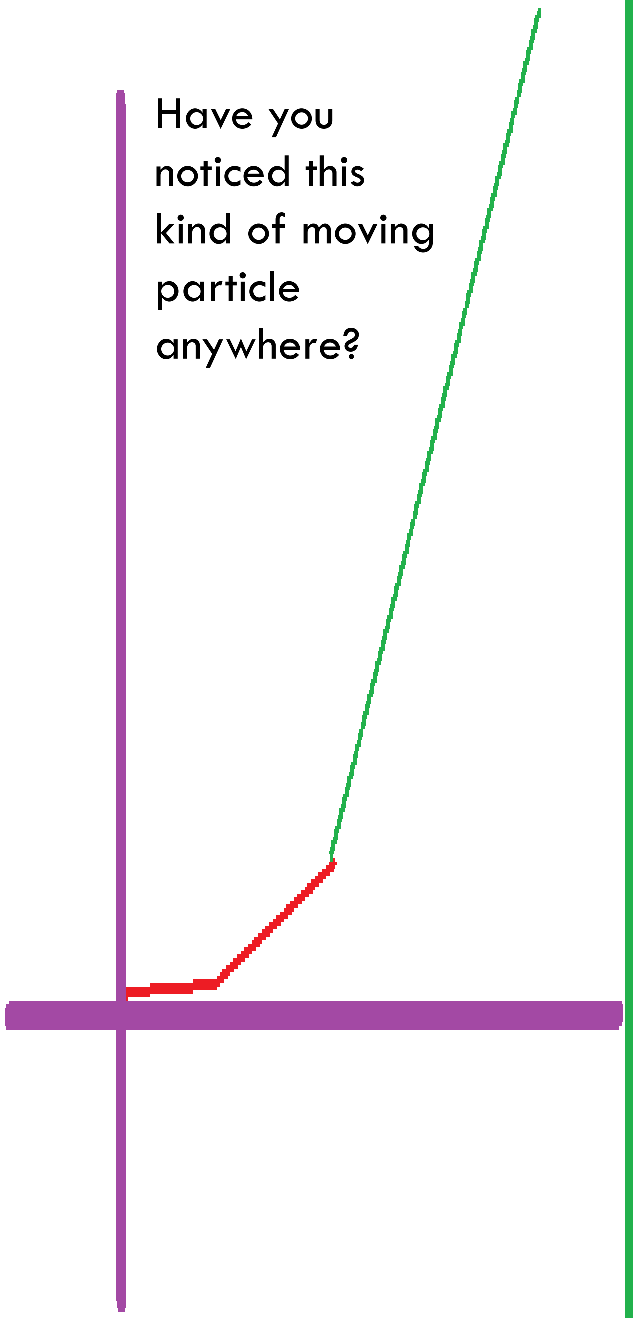
$$\Rightarrow x = \pm 4$$

When, $x = 4, y = \frac{4^3 + 2}{6} = \frac{66}{6} = 11$

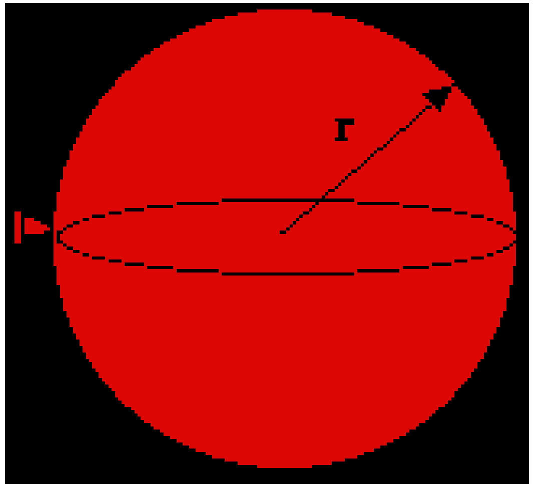
When,

$x = -4, y = \frac{(-4)^3 + 2}{6} = \frac{62}{6} = \frac{31}{3}$

Hence, the points required on the curve are $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$.



A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.



Solution

The volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^2$$

Diameter = $\frac{3}{2}(2x+1)$

$$r = \frac{3}{4}(2x+1)$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3 = \frac{9}{16}\pi(2x+1)^3$$

Hence, the rate of change of volume with respect to x is as

$$\frac{dV}{dt} = \frac{9}{16}\pi\frac{d}{dt}(2x+1)^3 = \frac{9}{16}\pi \text{x} 3(2x+1)^2 \text{ x } 2 = \frac{27}{8}\pi(2x+1)^3.$$

Question

The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced.

Solution

Marginal cost is the rate of change of total cost with respect to output.

$$\therefore \text{ Marginal cost } (MC) = \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$$

$$= 0.021x^2 - 0.006x + 15$$

When $x = 17$, $MC = 0.021(17^2) - 0.006(17) + 15$

$$= 0.021(289) - 0.006(17) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, when 17 units are produced, the marginal cost is Rs. 20.967.

MARGINAL COST IS THE FIRST DERIVATIVE OF THE TOTAL COST w.r.t quatity

Question

Sand is pouring from a pipe at the rate of $12\,cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

r = 6h

Solution

The volume of a cone (V) with radius (r) and height (h) is given by.

$$V = \frac{1}{3}\pi r^2 h$$

It is given that,

$$h = \frac{1}{6}r \Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$$

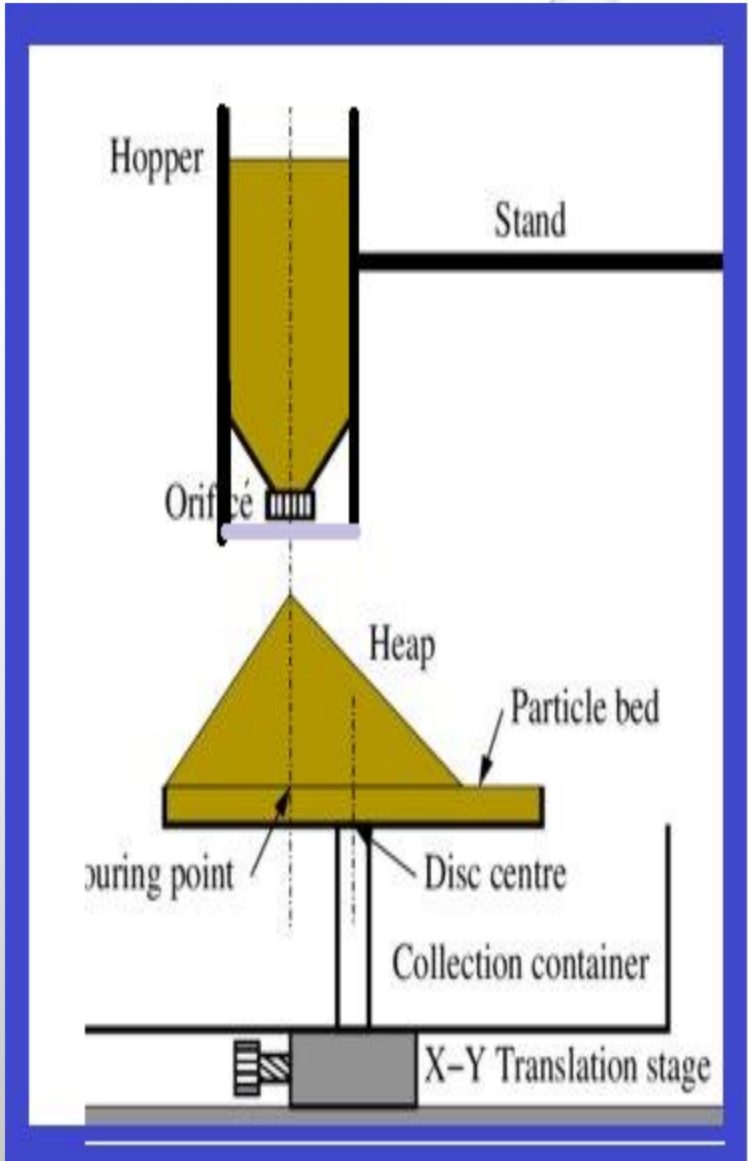
The rate of change of volume with respect to time (t) is

$$\frac{dV}{dt} = 12\pi\frac{d}{dh}(h^3).\frac{dh}{dt}$$

$$= 12\pi(3h^2)\frac{dh}{dt}$$

$$= 36\pi h^2\frac{dh}{dt}$$

It is also given that $\frac{dV}{dt} = 12\,cm^2/s$



Example 43

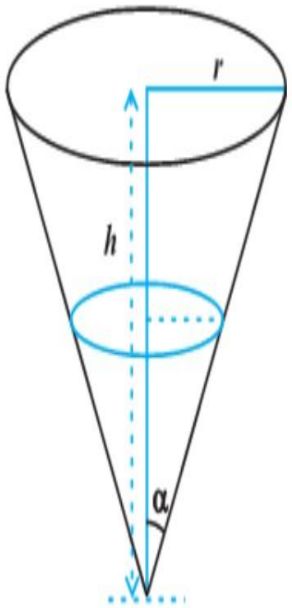
A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic meter per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

Water tank is in shape of cone

Let r be the radius of cone,

h be the height of cone,

& α be the semi-vertical angle



Given $\alpha = \tan^{-1}(0.5)$

So, $\tan\alpha = (0.5)$

$$\frac{r}{h} = 0.5$$

teachoo.com

$$\frac{r}{h} = \frac{1}{2}$$

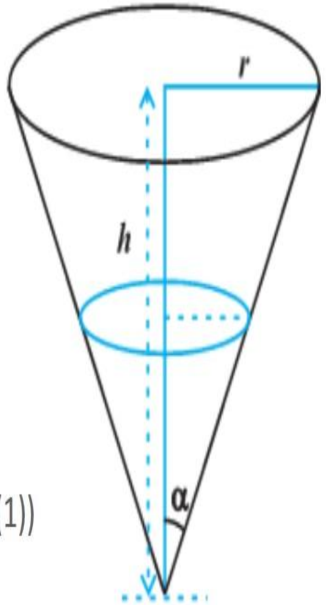
$$r = \frac{h}{2} \qquad ...(1)$$

Also,

Water is poured at a constant rate of 5 cubic meter per hour

So, $\frac{dV}{dt} = 5\, m^3/hr \quad ...(2)$

Where V is volume of cone



Now,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \quad \left(\text{As } r = \frac{h}{2} \text{ from (1)}\right)$$

$$dv/dt = ¼\, \pi h^2$$

teachoo.com

$$\frac{dh}{dt} = \frac{20}{\pi h^2} \qquad ...(3)$$

We need to find,

rate at which level of water is rising when depth is 4 m

i.e. $\left.\frac{dh}{dt}\right|_{h = 4m}$

Putting h = 4m in (3)

$$\left.\frac{dh}{dt}\right|_{h = 4m} = \frac{20}{\pi(h)^2} = \frac{20}{16} \times \frac{1}{\pi} = \frac{5}{4} \times \frac{1}{\frac{22}{7}} = \frac{5}{4} \times \frac{7}{22} = \frac{35}{88}$$

Hence, rate of change of water level is $\frac{35}{88}$ **m/hr.**

APPLICATION OF DERIVATIVES

TANGENT& NORMAL OF A PARABOLIC CURVE ON THE WAY TO HOME A RAINY DAY

MODULE 2

SLOPE

Find the slope of the tangents to the curve $y=\frac{x-1}{x-2}$, $x\neq 2$ at $x=10$.

The given curve is $y=\frac{x-1}{x-2}$.

$$\therefore \frac{dx}{dy}=\frac{(x-2)(1)-(x-1)(1)}{(x-2)^2}$$

$$=\frac{x-2-x+1}{(x-2)^2}=\frac{-1}{(x-2)^2}$$

Thus, the slope of the tangent at $x=10$ is given by,

$$\left.\frac{dx}{dy}\right]_{x=10}=\left.\frac{-1}{(x-2)^2}\right]_{x=10}=\frac{-1}{(10-2)^2}=\frac{-1}{64}.$$

Hence, the slope of the tangent at $x=10$ is $\frac{-1}{64}$.

Slope is $\frac{dy}{dx}$ at $(x_{1,}y_1)$

Parametric form

Question

Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution

It is given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

$$\therefore \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\therefore \frac{dx}{dy} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} == \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

Slope of normal = $-\frac{1}{\frac{dy}{dx}}$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dx}{dy}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}} = -\tan\frac{\pi}{4} = -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

FIND POINTS

Question

Find the points at which tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x - axis.

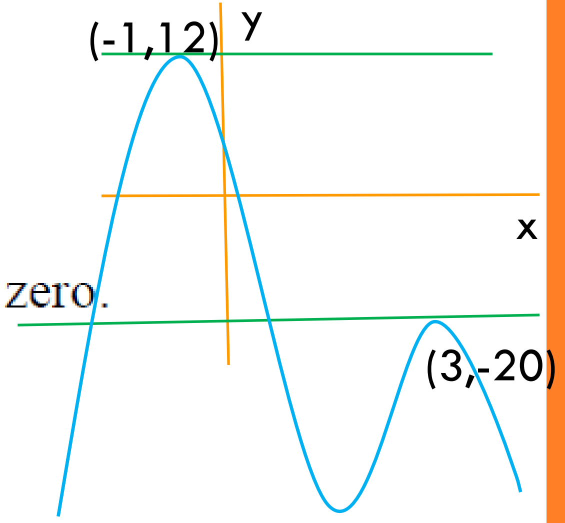
Solution

The equation of the given curve is $y = x^3 - 3x^2 - 9x + 7$.

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Slope of x axis=
tan 0 = 0

Now, the tangent is parallel to the x - axis if the slope of the tangent is zero.



$$\therefore 3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$$\Rightarrow x = 3 \ or \ x = -1$$

When $x = 3$, $y = (3)^3 - 9(3) + 7 = 27 - 27 - 27 + 7 = -20$.

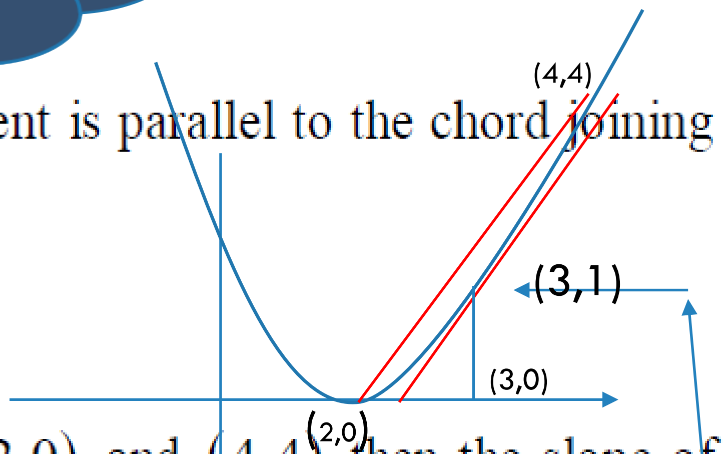
When, $x = 1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12$.

Hence, the points at which the tangent is parallel to the x-axis are $(3, -20)$ and $(-1, 12)$.

SLOPE OF LINE JOINING 2 POINTS$=\frac{y_2-y_1}{x_2-x_1}$

Question 8:

Find a point on the curve $y=(x-2)^2$ at which the tangent is parallel to the chord joining the points $(2,0)$ and $(4,4)$.



Solution 8:

If a tangent is parallel to the chord joining the points $(2,0)$ and $(4,4)$, then the slope of the tangent = the slope of the chord.

The slope of the chord is $\frac{4-0}{4-2}=\frac{4}{2}=2.$

Now, the slope of the tangent to the given curve at a point (x,y) is given by,

$$\frac{dy}{dx}=2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$2(x-2) = 2$

$x = 3$, then $y = (3-2)^2 = 1$

Hence the point is (3,1)

Question

Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) Perpendicular to the line $5y - 15x = 13$.

(a) The equation of the line is $2x - y + 9 = 0$

$2x - y + 9 = 0$, $y = 2x + 9$

This is of the form $y = mx + c$.

$\frac{dy}{dx} = m$

Slope of the line = 2

If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$2 = 2x - 2$

$\Rightarrow 2x = 4$

$\Rightarrow x = 2$

Now, $x = 2$

$\Rightarrow y = 4 - 4 + 7 = 7$

Thus, the equation of the equation of the tangent passing through $(2, 7)$ is given by,

$y - 7 = 2(x - 2)$

$\Rightarrow y - 2x - 3 = 0$

EQUATION OF STRAIGHT LINE(POINT –SLOPE FORM) $y - y_1 = m(x - x_1)$

(b) The equation of the line is $5y - 15x = 13$.

$5y - 15x = 13$, $y = 3x + \frac{13}{5}$

This is form of $y = mx + c$.

Slope isNegative reciprocal

Slope of the line = 3

If a tangent id perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is

$\frac{1}{\text{slope of the line}} = \frac{-1}{3}$.

$\Rightarrow 2x - 2 = \frac{-1}{3}$

Now, $x = \frac{5}{6}$

$\Rightarrow y = \frac{25}{36} + \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36} = \frac{217}{36}$

Thus, the equation of the tangent passing through $\left(\frac{5}{6}, \frac{217}{36}\right)$ is given by,

$y - \frac{217}{36} = \frac{1}{3}\left(x - \frac{5}{6}\right)$

$\Rightarrow \frac{36y - 217}{36} = \frac{-1}{18}(6x - 5)$

$\Rightarrow 36y - 217 = -2(6x - 5)$

$\Rightarrow 36y - 217 = -12x + 10$

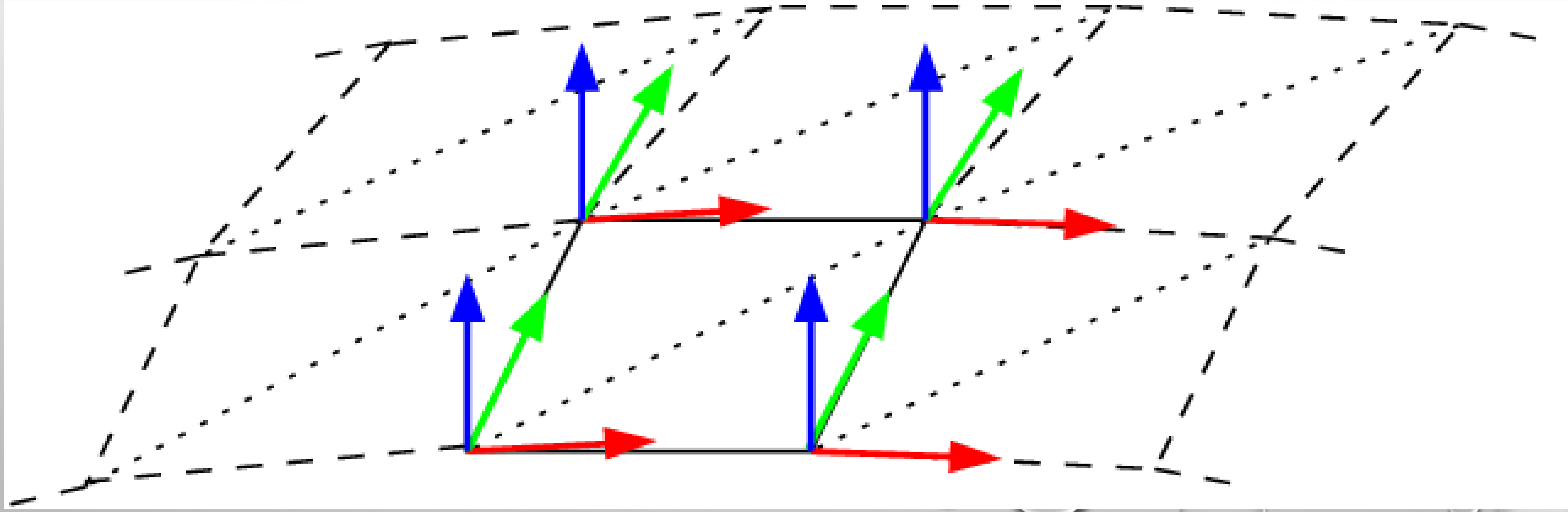
$\Rightarrow 36y + 12x - 227 = 0$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y - 15x = 13$) is $36y + 12x - 227 = 0$.

MODULE -3

TANGENTS AND NORMALS

IMPORTANT QUESTIONS



Question

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Solution :

The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point (x, y) is given by,

$$\left.\frac{dy}{dx}\right]_{(x,y)} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then $y = 3x^2$.

Also, we have $y = x^3$.

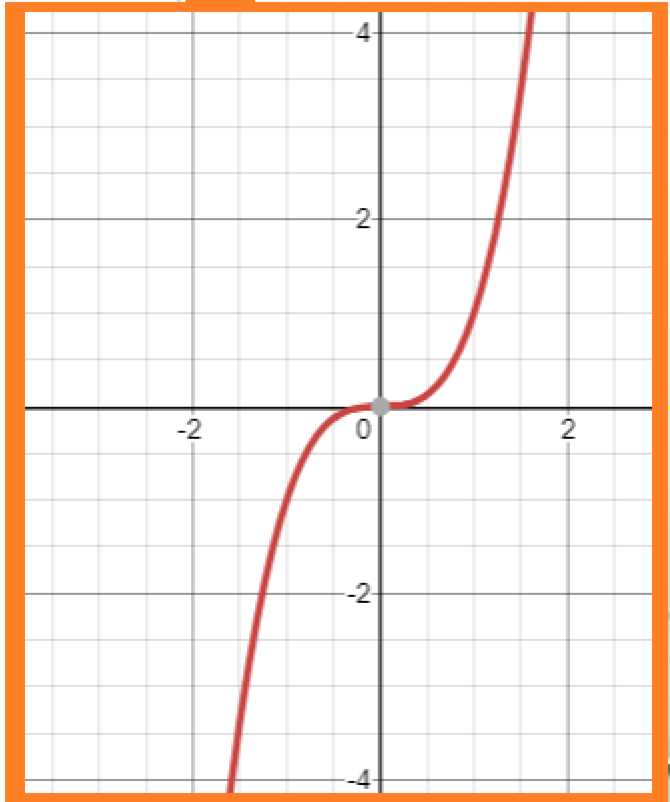
$$3x^2 = x^3$$

$$x^2(x-3) = 0$$

$$x = 0, x = 3$$

When $x = 0$, then $y = 0$ and when $x = 3$ then $y = 3(3)^2 = 27$.

Hence, the required points are $(0,0)$ and $(3,27)$.



Question

For the curve $y=4x^3-2x^5$, find all the points at which the tangents passes through the origin.

Solution

The equation of the given curve is $y=4x^3-2x^5$.

$$\therefore \frac{dy}{dx}=12x^2-10x^4$$

Therefore, the slope of the tangent at a point (x,y) is $12x^2-10x^4$.

The equation of the tangent at (x,y) is given by,

$$Y-y=\left(12x^2-10x^4\right)(X-x) \quad \text{..... (1)}$$

When the tangent passes through the origin $(0,0)$, then $X=Y=0$.

Therefore, equation (1) reduces to:

$$-y=\left(12x^2-10x^4\right)(-x)$$

$$y=12x^3-10x^5$$

Also, we have $y=4x^3-2x^5$.

$$\therefore 12x^3-10x^5=4x^3-2x^5$$

$$\Rightarrow 8x^5-8x^3=0$$

$$\Rightarrow x^5-x^3=0$$

$$\Rightarrow x^3\left(x^2-1\right)=0$$

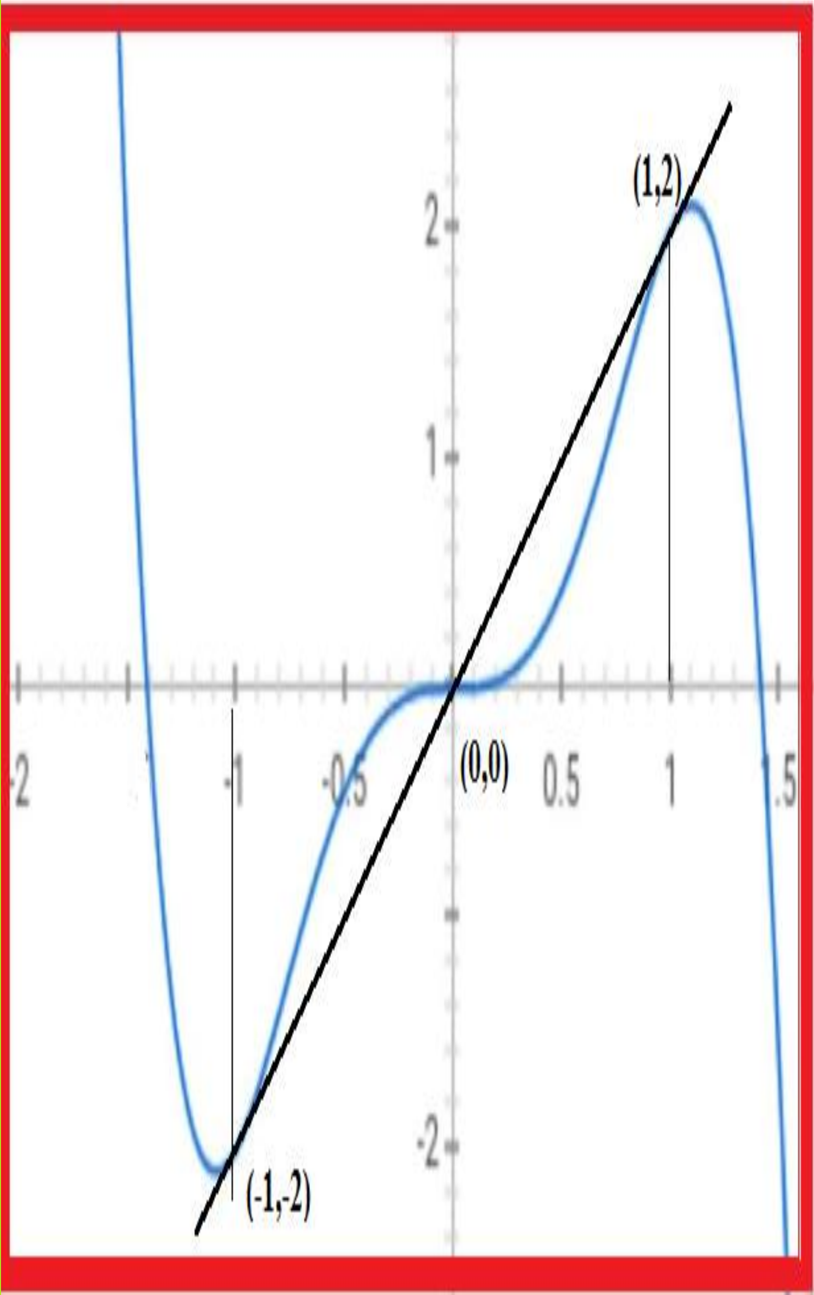
$$\Rightarrow x=0,\pm 1$$

When $x=0, y=4(0)^3-2(0)^5=0$.

When $x=1, y=4(1)^3-2(1)^5=2$.

When $x=-1, y=4(-1)^3-2(-1)^5=-2$.

Hence, the required points are $(0,0)$, $(1,2)$ and $(-1,-2)$.



Question

Find the equation of the tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0$.

Solution

The equation of the given curve is $y=\sqrt{3x-2}$.

The slope of the tangent to the given curve at any point (x,y) is given by,

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is $4x-2y+5=0$.

$4x-2y+5=0$, $y=2x+\frac{5}{2}$ (which is of the form $y=mx+c$

SQUARE ROOT FUNCTION

Slope of the line = 2

Now, the tangent to the given curve is parallel to the line $4x-2y-5=0$ if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x-2}}=2$$

$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$

$$\Rightarrow 3x-2=\frac{9}{16}$$

$$\Rightarrow x=\frac{41}{48}$$

When $x=\frac{41}{48}, y=\sqrt{3\left(\frac{41}{48}\right)-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$

Equation of the tangent passing through the point $\left(\frac{41}{48},\frac{3}{4}\right)$ is

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3=\left(\frac{48x-41}{6}\right)$$

$$\Rightarrow 24y-18=48x-41$$

$$\Rightarrow 48x-24y=23$$

Hence, the equation of the required tangent is $48x-24y=23$.

Question

Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}, x \neq 3$.

Solution

The equation of the given curve is $y = \frac{1}{x-3}, x \neq 3$

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-3)^2}$$

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$$\Rightarrow 2(x-3)^2 = -1$$

$$\Rightarrow (x-3)^2 = \frac{-1}{2}$$

This is not possible since the **L.H.S.** is positive while the **R.H.S.** is negative. Hence, there is no tangent to the given curve having slope 2.

NO TANGENTS WITH THE GIVEN CONDITIONS

Question :

Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Solution

The equation of the given curve is $y = x^3 - 11x + 5$.

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$).

$\therefore$ Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point (x, y) is given by,

$$\frac{dy}{dx} = 3x^2 - 11$$

Then, we have:

$$3x^2 - 11 = 1$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

When $x = 2, y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9$.

When $x = -2, y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19$.

Hence, the required points are $(2, -9)$??????

(-2,-19) REJECTED
WHY????